

The Digital Territory: a mathematical model of the concept and its properties

A. Kameas^{1,3} Y.C. Stamatou^{2,3}

¹Hellenic Open University, School of Sciences and Technology, Patras, Greece. e-mail: kameas@eap.gr

²University of Ioannina, Department of Mathematics, 451 10, Ioannina, Greece. e-mail: istamat@cc.uoi.gr

³Research Academic Computer Technology Institute, N. Kazantzaki, University of Patras, 26500, Rio, Patras, Greece

ABSTRACT

In this paper we attempt to define a mathematical model for the formal study of the concept of Digital Territories (DT) and its properties. We describe a simple DT model and employ the first order language of graphs and random graph properties in order to define and study properties of DTs that emerge as soon as the entities (within the DT) population reaches certain threshold points. Our aim is to show that the DT concept can be formalized and studied using mathematical tools that are within reach of any researcher in the DT domain in order to describe and study properties of interest.

1 Introduction

The concept of creating *artificial communities*, i.e. worlds composed of simulated, non-living entities, is a relatively old one and seems to have started with the development of the concept of *Artificial Life*. The concept of Artificial Life refers to the creation and simulation of populations composed of non-living organisms whose actions are simulated by special simulation programs [4]. The discipline “Artificial Life” differs from the discipline “Artificial Intelligence” in at least two things: (i) the former discipline is concerned with properties related to a *population* as a result of collective activity with the appearance of “social” behavior similar to human communities while the latter term is focused on an *individual* and how the individual may display abilities of a human quality, and (ii) the former discipline does not require any advanced intelligence from an organism while the latter discipline is targeted at achieving intelligence.

Nowadays, it seems that we are close to the development of the foundations of yet another “Artificial” concept: the *Digital Territory (DT)*. This discipline is of a totally different nature from both Artificial Life and Artificial Intelligence and it is based on new mathematical tools and more complex technological advancements. In a few words, the concept of a *Digital Territory* seems to integrate Artificial Life with Artificial Intelligence: it describes worlds with moving agents which, however, move in complex terrains which contain elements of both the physical and digital world (as opposed to organisms living within a computer simulation program) as well as “real” intelligence since it integrates devices with human beings in a complex pattern of interactions. An excellent introduction to Digital Territories can be found in [1].

In what follows, we will briefly introduce the Digital Territory concept and then propose a simple mathematical model in order to facilitate the study of its properties. We hope that our work will be a first step into formalizing the DT concept and uncovering, at the same time, directions for further foundational work in this new discipline.

2 A model for the DT concept

Ambient Intelligence (AmI) space consists of a set of technologies, infrastructures, applications and services operating seamlessly across physical environments (e.g. neighbourhood, home, car); thus spanning all the different spheres of everyday life. Physical space becomes augmented with digital content, thus transcending the limits of nature and of direct human perception. A new term is needed for this new kind of space, which captures its dual nature. The term “digital territory” (DT) has been coined in an attempt to port a real world metaphor into the forthcoming synthetic world [1].

2.1 DT basics

A DT is an ephemeral AmI space: it is created for a specific purpose and integrates the will of the owner (an individual or group operator) with the means to achieve it (including infrastructure, properties, services and objects) within an AmI space. A DT can be composed of sub-spaces, which are determined with respect to their services, usage, etc. For example, a DT could be distinguished in compartments, as in real life: private compartments (no interaction) and public compartments (where interaction with other DTs is possible). DTs form larger communities that can be thought of as DTs themselves since they have the defining characteristics given above. Thus, a DT may contain other DTs or be at the same level (i.e. independent). There’s also a need to define “mechanisms” to connect existing sub-DTs, e.g. like city districts having detached family housing areas, which have roads connecting separate family houses and their driveways to each other. Other interesting factors would be the critical mass and drift attributes, which concern sizes of groups and how uniform they grow or break apart into new individual DTs because of growth or hierarchical changes. However, as it is difficult to suggest that any DT is only of a digital nature, it would be actually problematic to try to separate digital and physical instantiations. One illustrative example of DTs is the home, which is conceived as a private place and created to protect the family. Services have to do with practical issues (like food and cleanliness) and with

relationships to the world (for example entertaining and entertainment). Size really depends on the identity of the individual and their current situation. A DT is defined not by physical space or borders alone but by the following non-physical characteristics as well: (i) information processing and/or information exchange with the external to the DT world, (ii) reaction to events taking place externally to the DT, (iii) its owner/governor, who uniquely characterizes this DT. However, since information is not a physical entity that can be enclosed within a clearly defined physical space (e.g. disk surface) but it can move and change forms and shapes (carrying the same information content semantically and entropically) it can be thought of as an ubiquitous entity possible shared by many DTs. Thus, an important characteristic of DTs is the ability to share, transform and route information (both about themselves and about other DTs).

2.2 The model

In order to model a DT we make the following simplifications and abstractions:

A *Digital Territory* is a spherical region in some d -dimensional space (e.g. 2-dimensional or 3-dimensional Euclidean spheres) composed of moving entities which appear or disappear in an unpredictable manner (uniformly at random) within the region. Around each of the entities another spherical region is defined, *the bubble*, composed of the space points lying within a prespecified distance from the entity (this models the sensory and communication constraints of the entity).

To enhance model generality, we define space as the set of values of a given property; then distance can be defined as a function from this set to the set of real numbers. In this way, we may define bubbles based on relations between values of any selected property and not only using spatial concepts.

We are interested in describing the activities of the entities within this region using a mathematical formalism and mapping the elements of the formalism into currently available technological devices.

Although the model described above incorporates the basic concepts of a DT it, nevertheless, cannot easily model some vital DT issues such as *security*. However, some security aspects that can be mathematically described using a special logic described in Section 3 can still be studied theoretically under the proposed model (see [3] on the key management issue for sensor networks).

3 The mathematics of the DT model

3.1 The fixed radius random graph model

We will now use a model, called the *fixed radius random graph model* and denoted by $G_{n,R_0,d}$, which can model the appearance and movement of DT entities, the bubbles, within a certain region as well as their pair-wise interactions.

According to this model, n points are generated uniformly at random in some d -dimensional metric space and an edge is drawn between two points only if their distance is at most R_0 (this models the fact that DT entities do not have infinite communication and sensory capabilities). With regard to the model $G_{n,R_0,d}$, in practical situations the metric space is either the 2-dimensional or 3-dimensional Euclidean space. In these spaces, distance is most often given through the l_2 Euclidean norm. In [2] the interested reader may find a number of theoretical results with regard to the properties of the fixed radius random graph model.

3.2 The first order language of graphs

We will now describe a formal language which allows us to describe and reason systematically about a number of statements referring to the relationships between nodes of any graph model although our focus will be the fixed radius random graph model. Our target is to show that many of these properties are certain to appear as a region's population increases and, thus, we can "predict" the emergence of certain properties when a DT starts to be heavily populated.

We will be confined to properties expressible in the *first order language* of graphs. This language can be used to describe some useful (and naturally occurring in applications) graph properties using elements of the first order logic.

The alphabet of the first order language of graphs consists of the following (see, e.g., [5]):

- Infinite number of variable symbols, e.g. $z,w,y...$ which represent graph vertices.
- The binary relations "==" (equality between graph vertices) and "~" (adjacency of graph vertices) which can relate only variable symbols, e.g. " $x\sim y$ " means that the graph vertices represented by the variable symbols x,y are adjacent.
- Universal, \exists , and existential, \forall , quantifiers (applied only to *singletons* of variable symbols).
- The Boolean connectives used in propositional logic, i.e. $\vee, \wedge, \neg, \dots$.

An example of graph property expressible in the first order language of graphs is the existence of a triangle: $\exists x \exists y \exists w (x \sim y) \wedge (y \sim w) \wedge (w \sim x)$. Another property is that the diameter of the graph is at most 2 (can be easily written for any fixed value k instead of 2): $\forall x \forall y [x = y \vee x \sim y \vee \exists w (x \sim w \wedge w \sim y)]$. However, other equally important graph properties, like connectivity, cannot be expressed in this language.

We will now define the important *extension statement* in natural language, although it clearly can be written using the first order language of graphs (see [5] for the details):

[Extension statement $A_{r,s}$] The extension statement $A_{r,s}$, for given values of r,s , states that for all distinct x_1, x_2, \dots, x_r and y_1, y_2, \dots, y_s there exists distinct z adjacent to all x_i s but no y_j .

The importance of the extension statement $A_{r,s}$ lies in the following. When applied to the first order language of graphs, if $A_{r,s}$ (for *all* r,s) holds for a random graph G (in some random graph model) with probability tending to 1 asymptotically with the number of vertices of the graph, then for every statement A written in the first order language of graphs either $\lim_{n \rightarrow \infty} \Pr[G(n,p) \text{ has } A] = 0$ or $\lim_{n \rightarrow \infty} \Pr[G(n,p) \text{ has } A] = 1$.

It can be proved, using results related to the distance between randomly chosen points in Euclidean spaces which are stated in Section 3.3, that the extension statement $A_{r,s}$ holds, almost certainly, for all r,s for a random graph according to the fixed radius model. Thus, all properties describing relationships between the nodes which expressible in the first order language of graphs are bound to appear almost certainly as the number of nodes (i.e. region's population) increases.

3.3 Random points in d -dimensional spheres and emergent properties

With regard to the distribution of the distance between points chosen uniformly at random to lie within a Euclidean d -dimensional sphere, the work in [6] derives the probability density function for the distance between two points. In order keep the exposition simple, we will confine ourselves to the 2-dimensional Euclidean sphere, i.e. the circle.

For randomly chosen points within a circle of radius R , the following is a direct consequence of the work in [6]:

Corollary 1 The distance between randomly chosen points within a circle of radius R obeys the following probability density and distribution functions:

$$P_2(s) = 2s/R^2 - (s^2/\pi R^4) \sqrt{(4R^2 - s^2)} - (4s/\pi R^2) \arccos(s/2R),$$

and

$$D_2(x) = 4(x/2R)^2 + (1/2\pi)(x/(2R))(4 - 4(x/2R)^2)^{3/2} - (3/\pi)(x/2R) \sqrt{(4 - 4(x/2R)^2)} + (2/\pi) \arcsin(x/2R) - (8/\pi)(x/2R)^2 \arccos x/2R. \quad (1)$$

The connection of the distribution function $D_2(x)$ for the distance between random points on the circle of radius R and the probability that the extension statement is given by the following lemma (see [3]):

Lemma 1 For the 2-dimensional sphere (circle) the probability that $A_{s,t}$ fails for $G_{n,R_0,d}$ is bounded from above as follows:

$$\Pr[A_{s,t} \text{ fails}] \leq n^{s+t} [1 - D_2(R_0)^s (1 - D_2(R_0))^t]^{n-(s+t)}. \quad (2)$$

Setting $\sigma = R_0/2R$, the following can be proved (see [3]):

Theorem 1. If $\sigma = R_0/2R = c$ is a constant, $0 < c < 1$, then Equation (2) tends to 0. If $\sigma = R_0/2R = f(n) = \omega(1/\sqrt{(n)})$, then Equation (2) also tends to 0.

The generalization to d -dimensional spheres is the following:

Theorem 2. Let $\sigma = R_0/2R = c$ be a constant, $0 < c < 1$. Then for any first order property A , then $\Pr[G_{n,R_0,d} \text{ has } A]$ tends to 1 or 0. If $\sigma = R_0/2R = f(n) = \omega(1/d \sqrt{(n)})$, then $\Pr[G_{n,R_0,d} \text{ has } A]$ tends to 1 or 0 too.

Considering the 2-dimensional case, what Theorem 1 states is that as the population size varies beyond a threshold (which is a function of the interaction radius), the extension statements start to hold with probability 1 which, implies, that any property written in the first order language of graphs holds either with probability tending to 0 or with probability tending to 1 (asymptotically, with the population size).

4 Interpretation and critique of the model within the DT domain

The core element of a Digital Territory that the model proposed in Subsection 2.2 tries to capture is the interaction between the DT entities (bubbles) and the way properties of the DT as a whole emerge as a result of these interactions. The interaction (through communication or external stimulation of entities' sensors) is captured by the fixed radius random graph model and the radius parameter R_0 . The fact that properties emerge or disappear as the number of entities varies (as well as the radius R_0) is captured by the

asymptotic validity with probability 0 or 1 of the extension statement.

This model can be employed in the following way in the study of DTs:

1. We first define the radius of interaction so as to capture the interaction ranges of the studied DT entities.
2. We then try to write the property of interest in the first order language of graphs.
3. Then we study the relationship between the number of the entities and the radius of interaction using Theorem 1 in order to prove that the property almost certainly holds or almost certainly does not hold as the entity population grows larger. Which of the cases is true will depend on whether the property is *monotone* i.e. whether it is true that if it holds for a certain population size then it certainly holds or does not hold when the population increases.

Let us consider an example using the following property $P(k)$:

$P(k)$ = the diameter of the fixed radius random graph is at most k . The diameter of a graph is defined as the longest path between two nodes of the graph.

This property can be written in the first order language of graphs as follows:

$$\forall x \forall y [x \neq y \vee x \sim y \vee \exists w_1 \exists w_2 \dots \exists w_{k-2} (\wedge_{i,j \in [1 \dots k-2]}: i \neq j \wedge w_i \neq w_j \wedge x \sim w_1 \wedge w_1 \sim w_2 \wedge w_2 \sim w_3 \dots \wedge w_{k-2} \sim y)].$$

Since this property is monotone decreasing, i.e. if it holds for a given graph then it is not certain that it will hold for a graph containing additional edges, and we know that this property has to hold with probability 0 or 1 (asymptotically with the population size), then the property holds with probability 0 for sufficiently dense graphs. Sufficiently dense means either many entities within a fixed radius DT or long interaction ranges.

The property $P(k)$ has the interpretation that in order for a message to be publicly known among all DT entities, it does not need to “visit” more than k entities. This may model, for instance, the spread of a virus in cellular phone network region cell (“DT”) composed of roaming users with mobile phones (“bubbles”): many users or long wireless connection ranges (of the mobile phones) imply fast virus spread.

5 Conclusions

In this paper we have made a first step towards the development of a mathematical model for the study of

the DT concept and its properties. We believe that the rich mathematical theory of the first order language of graphs and threshold phenomena can be used for defining basic DT properties and explain their emergence or disappearance as a function of the DT population size and communication/sensory capabilities of the population entities.

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